Munch

March 19, 2001

Appetizer

[some pictures of a triangle eaten by disks and of a *disk* eaten by *triangles*...]

General abstract rules

Before the game starts, players get together and fix the space in which they are going to have a dinner, equip the space with translations, fix a set X in it that shall be the common shape of their teeth, fix a point c in X that shall help aiming, fix a set S_0 in the space that shall be the pie to eat, and fix in which order they are going to munch at the pie.

Then, in move $i \in \{1, 2, 3, \dots\}$, the player to move selects a point $z_i \in S_{i-1}$ still in the pie, aims her teeth at z_i and bites off an X-shaped piece of the pie so that the next player is left with $S_i := S_{i-1} \smallsetminus (X$ shifted with c matching $z_i)$.

In the game *Munch Last* she wins if she eats the last bits of the pie, that is if $S_i = \varnothing$ after her move. In the game *Don't Munch Last* she loses under the same condition.

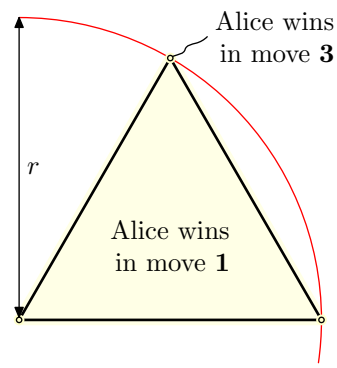
To be sure the game always ends, we may further assume that the space is equipped with a distance, that c is at the center of a (small) ball (of radius r, let's say) entirely included in X, that S_0 is bounded, and that the space is classical enough to prevent infinitely many points at mutual distance $\geq r$ to lay all in a bounded set.

Munch Last at a triangle with disks

Here we consider two alternating players Alice and Bob eating $S_0 :=$ a filled closed equilateral triangle of edge length 1 in the Euclidean plane $\mathbb{R}^2$ with the teeth $X :=$ the open disk $\{z \in \mathbb{R}^2 : |z| < r\}$ of fixed radius $r > 0$ and centered on $c = (0, 0)$. A move at a point z_i results in removing all points at distance $< r$ from z_i.

When $r > 1$, Alice wins in move 1 wherever she plays, because one disk is big enough to remove anything from the triangle wherever the center lays in the triangle. When $r = 1$, the same happens except on the vertices of the triangle

where Alice would win only in move 3 because we decided the disk X doesn't include its boundary. As r gets smaller, new areas appear that contain losing first moves for Alice. If Alice plays there, Bob can then make the game ends after an even total number of moves, however Alice defends herself.



$r = 1$

This game is the same as packing non-overlapping disks of radius $r/2$ inside an equilateral triangle of edge length $1 + r\sqrt{3}$.

[some pictures...]

Finding the best move

Consider the general rules again with only two alternating players. In general the player to move is facing a set S of points remaining from the initial pie and wants to know which point is best to select as a move.

Define $F_j(S) :=$ the subset of points z in S such that the game shall be over after j more moves, provided the very next move is z and the two players then play perfectly well. Here *playing perfectly well* means *winning as soon as possible* if there is a way to win and *losing as late as possible* otherwise. Hence

$$\begin{aligned}
F_1(S) &= \text{the points } z \text{ in } S \text{ that leave } S \text{ empty after removing } X(z), \\
F_2(S) &= \text{the other points } z \text{ in } S \text{ such that } F_1(S \smallsetminus X(z)) \text{ is not empty}, \\
F_3(S) &= \text{the remaining other points such that } F_2(S \smallsetminus X(z)) \text{ is not empty}, \\
&\vdots
\end{aligned}$$

where $X(z)$ denotes the set X shifted so that c matches z. In other words

$$\begin{aligned}
F_1(S) &= \{z \in S : S \subseteq X(z)\} \\
F_2(S) &= \{z \in S : F_1(S \smallsetminus X(z)) \neq \varnothing\} \smallsetminus F_1(S) \\
F_3(S) &= \{z \in S : F_2(S \smallsetminus X(z)) \neq \varnothing\} \smallsetminus (F_1(S) \cup F_2(S))
\end{aligned}$$

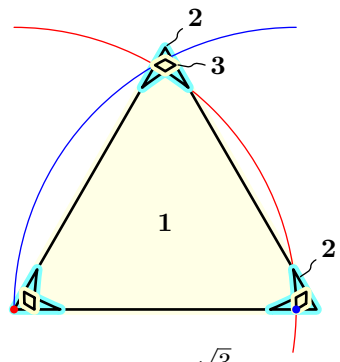
$$\vdots$$

Well, it does not seem easy to solve these formulas even in particular cases.

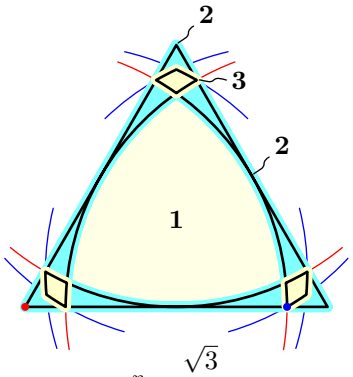
Solving 'Tridisk'

1 means Alice wins in move 1
2 means Alice loses in move 2
3 means Alice wins in move 3
4 means Alice loses in move 4

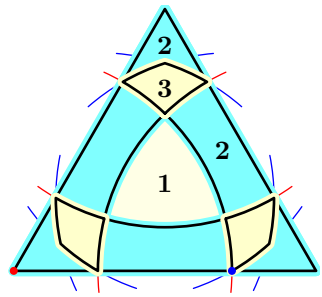
$$\vdots$$



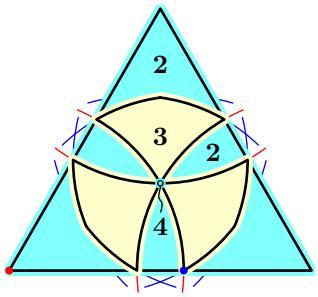
$1 > r > \frac{\sqrt{3}}{2}$



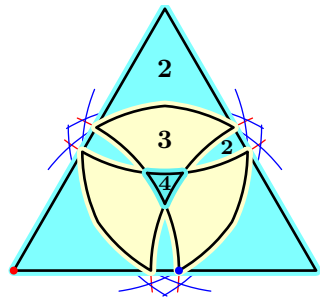
$r = \frac{\sqrt{3}}{2}$



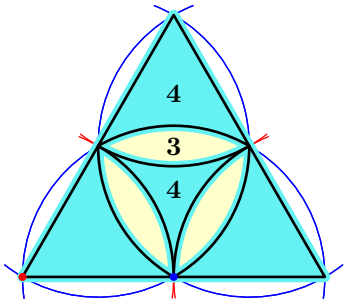
$\frac{\sqrt{3}}{2} > r > \frac{\sqrt{3}}{3}$



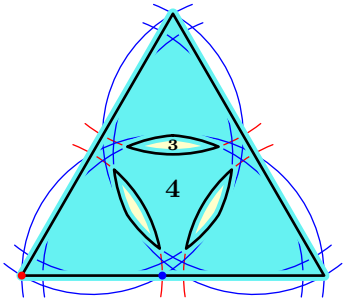
$$r = \frac{\sqrt{3}}{3}$$



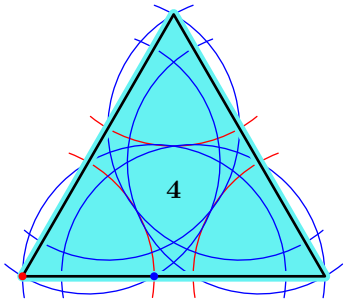
$$\frac{\sqrt{3}}{3} > r > \frac{1}{2}$$



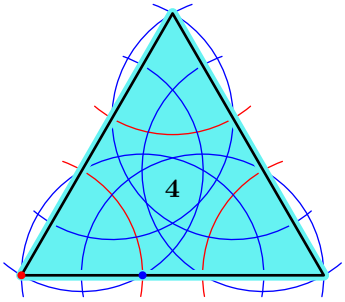
$$r = \frac{1}{2}$$



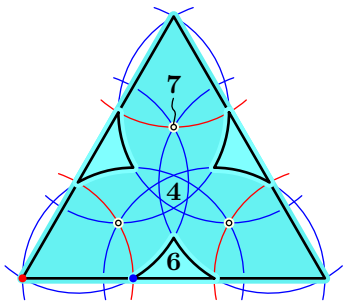
$$\frac{1}{2} > r > \frac{1+\sqrt{3}-\sqrt{2\sqrt{3}}}{2}$$



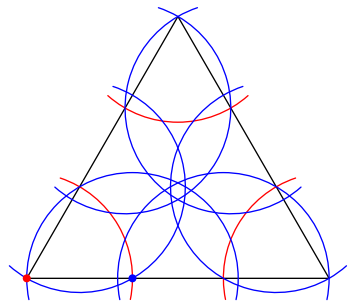
$$r = \frac{1+\sqrt{3}-\sqrt{2\sqrt{3}}}{2}$$



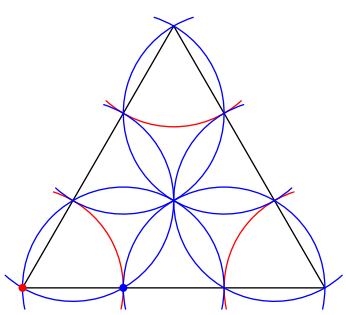
$$\frac{1+\sqrt{3}-\sqrt{2\sqrt{3}}}{2} > r > \frac{\sqrt{3}-1}{2}$$



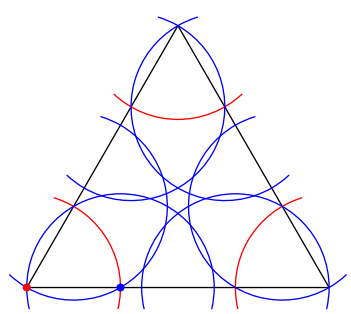
$$r = \frac{\sqrt{3}-1}{2}$$



$$\frac{\sqrt{3}-1}{2} > r > \frac{1}{3}$$



$$r = \frac{1}{3}$$



$$r = .31$$